

A 4-SPHERE WITH NON CENTRAL RADIUS AND ITS INSTANTON SHEAF

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ABSTRACT. We build an $SU(2)$ -Hopf bundle over a quantum toric four-sphere whose radius is non central. The construction is carried out using local methods in terms of sheaves of Hopf-Galois extensions. The associated instanton bundle is presented and endowed with a connection with anti-selfdual curvature.

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1. INTRODUCTION

Gauge theories are one of the most beautiful example of the fruitful interplay between mathematics and physics. The approach to the study of instantons in terms of bundles and connections with (anti)selfdual curvatures goes back to the late 70's [1]. The basic case is that of $SU(2)$ -instantons on the four-sphere S^4 and their underlying Hopf bundle $S^7 \rightarrow S^4$.

In the last years various constructions of instanton bundles on noncommutative spheres have been provided by drawing on different methods accordingly to the 'kind' of non-commutativity into play (isospectral, quantum groups, twists,...).

In this paper we work over a noncommutative four sphere $\mathcal{A}(S_q^4)$ that was first introduced in [4] in the framework of 2-cocycle deformations of toric varieties. Noncommutative toric varieties belong to a vast class of noncommutative spaces obtained by using Drinfeld's general theory of twists (or 2-cocycles) as underlying source of noncommutative deformation. In the specific case treated there, the Hopf algebra of symmetries on which the 2-cocycle is based is the algebraic torus $(\mathbb{C}^\times)^n$. Twisting the symmetry induces a functorial deformation in the category of its (co)modules; namely, in every (co)algebra carrying a (co)action of $(\mathbb{C}^\times)^n$ there is a natural non(co)commutative (co)product whose explicit expression depends on the Drinfeld twist. The algebra $\mathcal{A}(S_q^4)$ and its local patches

we present below (and essentially every noncommutative algebra throughout the paper), fit within this machinery. Nevertheless we will not make explicit use of this formalism in the present paper and we refer the reader to [3, 4] for all the details relevant to these specific deformations.

A key feature of this sphere $\mathcal{A}(S_q^4)$ is the noncommutativity of the radius, i.e. the algebra generator playing the role of the radius does not belong to the center of the algebra. Related to this is also the fact that the sphere is more effectively described in terms of local charts (see §2 below). A main consequence of these peculiarities is that previous approaches adopted in the study of instantons on quantum spheres (for instance the construction of a globally defined instanton projector as in [6, 10]) cannot be used here. The problem is overcome by making use of sheaf theory. We look at $\mathcal{A}(S_q^4)$ as a ‘locally ringed quantum space’ and use the noncommutative sheaf-theoretical methods developed in [12] to assemble a noncommutative Hopf bundle on it. This later will consist in the data of a sheaf of Hopf-Galois extensions.

The paper is organized as follows. A first section (§2) is dedicated to recall the 4-sphere $\mathcal{A}(S_q^4)$ with its description in terms of (isomorphic) ‘local patches’ ${}_N\mathbb{R}_q^4, {}_S\mathbb{R}_q^4$ obtained through noncommutative localization techniques. In §3 we give a revised characterization of $\mathcal{A}(S_q^4)$ (and its differential calculus) in terms of sheaves of quantum algebras on the classical sphere S^4 . Such a sheaf-theoretic description of quantum spaces, that well befits our $\mathcal{A}(S_q^4)$, is based on [12]. Crucial for our purpose is the factorization of the intersection of the two local charts ${}_N\mathbb{R}_q^4, {}_S\mathbb{R}_q^4$ as a twisted tensor product of the algebra $\mathcal{A} = \mathcal{A}(SU(2))$ of coordinate functions on $SU(2)$ and a 1-dimensional interval I . The two algebras \mathcal{A} and I are both commutative, but their twisted tensor product has a noncommutative algebra multiplication (cf. Prop. 3.7). This factorization is used in §4 to introduce ‘transition functions’ allowing for the reconstruction of a quantum principal $\mathcal{A}(SU(2))$ -bundle \mathcal{P} on $\mathcal{A}(S_q^4)$ out of trivial bundles on the local charts. The (noncommutative) algebraic language for principal bundles is that of Hopf-Galois extensions. We prove that the sheaf of the quantum principal $\mathcal{A}(SU(2))$ -bundle is a sheaf of Hopf-Galois extension. We conclude the section by constructing a sheaf of algebras associated to the fundamental (co)representation of $SU(2)$ on \mathbb{C}^2 , playing the role of the basic instanton bundle on $\mathcal{A}(S_q^4)$, and by sketching how to get instanton bundles with higher instanton numbers. In the last section, §5, we define an $su(2)$ -valued connection one form and prove that its curvature is anti-selfdual with respect to the Hodge $*$ -operator on the calculus on $\mathcal{A}(S_q^4)$. Appendix A contains a few remarks about $*$ -structures for twisted tensor products.

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2. THE QUANTUM 4-SPHERE AND ITS LOCAL CHARTS

We dedicate this section to recall the description of the noncommutative four-sphere $\mathcal{A}(S_q^4)$ as from [4, §3.5]. The algebra $\mathcal{A}(S_q^4)$ we are going to introduce is a one real parameter deformation of the algebra of coordinate functions on the classical four-sphere, the latter seen as a real subspace of the Klein quadric in \mathbb{CP}^5 . In [3, §5.3] the authors define for each couple of integers $d < n$ a noncommutative Grassmannian $\mathcal{A}(\text{Gr}_\theta(d, n))$ as a quotient of a projective space $\mathcal{A}(\mathbb{CP}_\theta^N)$, $N = \binom{n}{d} - 1$. The commutation relations among the generators of these algebras depend on an $n \times n$ matrix θ of parameters of deformation and are derived from a general theory of cocycle (or Drinfeld twists) Hopf deformations. For $d = 2$, $n = 4$ and a particular choice of the matrix θ which reduces the parameters from six complex θ_{ij} to a pure imaginary θ , or equivalently to a real one $q = \exp(i\theta/2) \in \mathbb{R}$, one obtains the noncommutative Grassmannian $\mathcal{A}(\text{Gr}_q(2, 4))$ as the algebra generated by ‘Plücker coordinates’ Λ_{ij} , $i < j$ ($i, j = 1, \dots, 4$), subject to the following commutation relations

$$\begin{aligned} \Lambda_{12}\Lambda_{13} &= q^{-2}\Lambda_{13}\Lambda_{12} \quad , \quad \Lambda_{12}\Lambda_{14} = q^{-2}\Lambda_{14}\Lambda_{12} \quad , \quad \Lambda_{12}\Lambda_{23} = q^2\Lambda_{23}\Lambda_{12} \quad , \\ \Lambda_{12}\Lambda_{24} &= q^2\Lambda_{24}\Lambda_{12} \quad , \quad \Lambda_{12}\Lambda_{34} = \Lambda_{34}\Lambda_{12} \quad , \quad \Lambda_{13}\Lambda_{14} = \Lambda_{14}\Lambda_{13} \quad , \\ \Lambda_{13}\Lambda_{23} &= q^2\Lambda_{23}\Lambda_{13} \quad , \quad \Lambda_{13}\Lambda_{24} = q^2\Lambda_{24}\Lambda_{13} \quad , \quad \Lambda_{13}\Lambda_{34} = \Lambda_{34}\Lambda_{13} \quad , \\ \Lambda_{14}\Lambda_{23} &= q^2\Lambda_{23}\Lambda_{14} \quad , \quad \Lambda_{14}\Lambda_{24} = q^2\Lambda_{24}\Lambda_{14} \quad , \quad \Lambda_{14}\Lambda_{34} = \Lambda_{34}\Lambda_{14} \quad , \\ \Lambda_{23}\Lambda_{24} &= \Lambda_{24}\Lambda_{23} \quad , \quad \Lambda_{23}\Lambda_{34} = \Lambda_{34}\Lambda_{23} \quad , \quad \Lambda_{24}\Lambda_{34} = \Lambda_{34}\Lambda_{24} \end{aligned} \quad (2.1)$$

and satisfying the Klein quadric identity

$$q\Lambda_{12}\Lambda_{34} - \Lambda_{13}\Lambda_{24} + \Lambda_{14}\Lambda_{23} = 0. \quad (2.2)$$

Further, we introduce a $*$ -structure on $\mathcal{A}(\text{Gr}_q(2, 4))$ by defining it on the generators as ¹

$$\Lambda_{12}^* = \Lambda_{12} \quad , \quad \Lambda_{13}^* = q\Lambda_{24} \quad , \quad \Lambda_{14}^* = -q\Lambda_{23} \quad , \quad \Lambda_{34}^* = \Lambda_{34} \quad (2.3)$$

and extending it to the whole algebra as an anti-algebra morphism. The quadric equation above has real form

$$q\Lambda_{12}\Lambda_{34} - q^{-1}\Lambda_{13}\Lambda_{13}^* - q^{-1}\Lambda_{14}\Lambda_{14}^* = 0 \quad (2.4)$$

which is shown to correspond to signature (5, 1) by considering the change of generators $X := \frac{1}{2}q(\Lambda_{12} - \Lambda_{34})$ and $R := \frac{1}{2}q(\Lambda_{12} + \Lambda_{34})$, thus giving

$$X^2 + \Lambda_{13}\Lambda_{13}^* + \Lambda_{14}\Lambda_{14}^* = R^2. \quad (2.5)$$

Definition 2.1. *The coordinate algebra $\mathcal{A}(S_q^4)$ is the $*$ -algebra generated by elements Λ_{ij} , $i < j$, $i, j = 1, \dots, 4$ satisfying the commutation relation (2.1) and the quadric identity (2.4), and with $*$ -structure as in (2.3).*

$\mathcal{A}(S_q^4)$ is a one-parameter deformation of a 4-sphere described in homogeneous coordinates. While the deformation parameter q enters the commutation relations (2.1) among the generators, the sphere relation (2.5) is classical. We remark that the ‘radius’ R of $\mathcal{A}(S_q^4)$ is noncommutative, namely it does not belong to the center of the algebra. Moreover R

¹The $*$ -structure in [4] was defined slightly differently with $\Lambda_{14}^* = -q^{-1}\Lambda_{23}$.

does not even generate a left or right denominator set (in the sense of Ore localization, see discussion just below), so that it is not possible to localize with respect to it.

We now introduce two ‘local patches’ ${}_N\mathbb{R}_q^4$, ${}_S\mathbb{R}_q^4$ of the sphere $\mathcal{A}(S_q^4)$ which are obtained via Ore localization with respect to the two real generators Λ_{34} and Λ_{12} respectively. Such localizations provide quantum analogue of stereographic projections from the North and South poles; we refer to the original paper [4] for details. Let us stress a substantial difference between the two generators: while Λ_{34} is central and hence the Ore localization reduces to the standard commutative localization, Λ_{12} is not.

Proposition 2.2. [4, Prop.3.19] *The degree zero subalgebra $(\mathcal{A}(\text{Gr}_\theta(d, n))[\Lambda_{34}^{-1}])_0$ of the **right** Ore localization of $\mathcal{A}(\text{Gr}_\theta(d, n))$ with respect to Λ_{34} is isomorphic to the algebra generated by elements β_{13} , β_{14} , β_{23} and β_{24} subject to the commutation relations*

$$\begin{aligned} \beta_{13}\beta_{14} &= \beta_{14}\beta_{13} , & \beta_{13}\beta_{23} &= q^2\beta_{23}\beta_{13} , & \beta_{13}\beta_{24} &= q^2\beta_{24}\beta_{13} \\ \beta_{14}\beta_{23} &= q^2\beta_{23}\beta_{14} , & \beta_{14}\beta_{24} &= q^2\beta_{24}\beta_{14} , & \beta_{23}\beta_{24} &= \beta_{24}\beta_{23} \end{aligned} \quad (2.6)$$

The proof (that we avoid to recopy here) is a direct computation once we make the identification $\beta_{ij} = \Lambda_{ij}\Lambda_{34}^{-1}$. Note that the generator $\beta := \Lambda_{12}\Lambda_{34}^{-1}$ is redundant, indeed the Plücker relation (2.2) implies

$$\beta = q(\beta_{24}\beta_{13} - \beta_{23}\beta_{14}) = q^{-1}(\beta_{13}\beta_{24} - \beta_{14}\beta_{23}). \quad (2.7)$$

The $*$ -structure defined in (2.3) induces on the generators β_{ij} the relations

$$\beta_{24}^* = q^{-1}\beta_{13} , \quad \beta_{23}^* = -q^{-1}\beta_{14} . \quad (2.8)$$

The resulting $*$ -algebra will be denoted by ${}_N\mathbb{R}_q^4$. The above equation (2.7) becomes

$$\beta = \beta_{13}^*\beta_{13} + \beta_{14}^*\beta_{14} = \beta_{23}^*\beta_{23} + \beta_{24}^*\beta_{24} . \quad (2.9)$$

Proposition 2.3. [4, Prop.3.22] *The degree zero subalgebra $([\Lambda_{12}^{-1}]\mathcal{A}(\text{Gr}_\theta(d, n)))_0$ of the **left** Ore localization of $\mathcal{A}(\text{Gr}_\theta(d, n))$ with respect to Λ_{12} is isomorphic to the algebra generated by elements α_{13} , α_{14} , α_{23} and α_{24} subject to the commutation relations*

$$\begin{aligned} \alpha_{13}\alpha_{14} &= \alpha_{14}\alpha_{13} , & \alpha_{13}\alpha_{23} &= q^{-2}\alpha_{23}\alpha_{13} , & \alpha_{13}\alpha_{24} &= q^{-2}\alpha_{24}\alpha_{13} \\ \alpha_{14}\alpha_{23} &= q^{-2}\alpha_{23}\alpha_{14} , & \alpha_{14}\alpha_{24} &= q^{-2}\alpha_{24}\alpha_{14} , & \alpha_{23}\alpha_{24} &= \alpha_{24}\alpha_{23} \end{aligned} \quad (2.10)$$

Similarly to the previous case, one gets the relations above by putting $\alpha_{ij} = \Lambda_{12}^{-1}\Lambda_{ij}$; by using (2.2) the element $\alpha := \Lambda_{12}^{-1}\Lambda_{34}$ can be expressed in terms of the α_{ij} as

$$\alpha = q^{-1}(\alpha_{24}\alpha_{13} - \alpha_{23}\alpha_{14}) = q(\alpha_{13}\alpha_{24} - \alpha_{14}\alpha_{23}) . \quad (2.11)$$

We denote by ${}_S\mathbb{R}_q^4$ the $*$ -algebra generated by the α_{ij} and $*$ -structure

$$\alpha_{24}^* = q^{-3}\alpha_{13} , \quad \alpha_{23}^* = -q^{-3}\alpha_{14} . \quad (2.12)$$

induced from (2.3). The equation (2.11) becomes

$$\alpha = q^{-4}(\alpha_{13}^*\alpha_{13} + \alpha_{14}^*\alpha_{14}) = q^4(\alpha_{23}^*\alpha_{23} + \alpha_{24}^*\alpha_{24}) .$$

Proposition 2.4. *There exists a $*$ -algebra isomorphism $Q : {}_S\mathbb{R}_q^4 \rightarrow {}_N\mathbb{R}_q^4$ defined on generators as*

$$Q(\alpha_{13}) = q^{-2}\beta_{13}, \quad Q(\alpha_{24}) = q^2\beta_{24}, \quad Q(\alpha_{14}) = q^{-2}\beta_{14}, \quad Q(\alpha_{23}) = q^2\beta_{23}, \quad Q(q) = q^{-1} . \quad (2.13)$$

The proof is a direct computation and we omit it. A geometric interpretation of this isomorphism is discussed in Remark 3.13.

The intersection of the two 'charts' ${}_N\mathbb{R}_q^4$ and ${}_S\mathbb{R}_q^4$ is geometrically obtained by removing the 'origin' in each patch. Algebraically this amounts to extend ${}_N\mathbb{R}_q^4$ with the inverse of β ; namely we introduce an extra generator β^{-1} with commutation relations

$$\beta_{13}\beta^{-1} = q^{-2}\beta^{-1}\beta_{13}, \quad \beta_{14}\beta^{-1} = q^{-2}\beta^{-1}\beta_{14}, \quad \beta_{23}\beta^{-1} = q^2\beta^{-1}\beta_{23}, \quad \beta_{24}\beta^{-1} = q^2\beta^{-1}\beta_{24} \quad (2.14)$$

together with

$$\beta^{-1}(\beta_{13}\beta_{24} - \beta_{14}\beta_{23}) = q = (\beta_{13}\beta_{24} - \beta_{14}\beta_{23})\beta^{-1}. \quad (2.15)$$

We denote ${}_{SN}\mathbb{R}_q^4$ the extension of ${}_N\mathbb{R}_q^4$ with respect to β^{-1} . Similarly we can extend ${}_S\mathbb{R}_q^4$ with the inverse of α , introducing an extra generator α^{-1} with commutation relations

$$\alpha_{13}\alpha^{-1} = q^2\alpha^{-1}\alpha_{13}, \quad \alpha_{14}\alpha^{-1} = q^2\alpha^{-1}\alpha_{14}, \quad \alpha_{23}\alpha^{-1} = q^{-2}\alpha^{-1}\alpha_{23}, \quad \alpha_{24}\alpha^{-1} = q^{-2}\alpha^{-1}\alpha_{24} \quad (2.16)$$

together with

$$\alpha^{-1}(\alpha_{24}\alpha_{13} - \alpha_{23}\alpha_{14}) = q = (\alpha_{24}\alpha_{13} - \alpha_{23}\alpha_{14})\alpha^{-1}. \quad (2.17)$$

We denote the resulting algebra ${}_{NS}\mathbb{R}_q^4$. The above two descriptions of the intersection are equivalent, as a consequence of Proposition 2.4:

Corollary 2.5. *The map Q extends to a $*$ -algebra isomorphism $Q : {}_{NS}\mathbb{R}_q^4 \rightarrow {}_{SN}\mathbb{R}_q^4$ with $Q(\alpha) = \beta$.*

An algebra isomorphisms between ${}_{NS}\mathbb{R}_q^4$ and ${}_{SN}\mathbb{R}_q^4$ was already observed in [4], but realized there via a different map, namely $G : {}_{NS}\mathbb{R}_q^4 \rightarrow {}_{SN}\mathbb{R}_q^4$ defined on generators as $G(\alpha_{ij}) = \alpha\beta_{ij}$, $G(\beta) = \alpha^{-1}$ and $G(q) = q$. The different choice of the isomorphism Q made here is relevant to what follows.

3. THE SHEAF OF THE QUANTUM 4-SPHERE

In this section we construct a sheaf of noncommutative algebras on the classical four-sphere S^4 . This is the structure sheaf describing a *quantum space* in the terminology of [12, §2], and it is obtained by considering suitable algebras constructed out of the local charts on the noncommutative 4-sphere $\mathcal{A}(S_q^4)$. We also provide an analogous sheaf-theoretic realization of the noncommutative differential calculus of $\mathcal{A}(S_q^4)$.

We begin with a description of the intersection of the charts in terms of *twisted tensor product* of algebras. We refer to Appendix A for terminology and the few details needed from the theory of twisted tensor products. The convenience of this approach will be manifest in the construction of the quantum Hopf bundle in §4.

3.1. The intersection of charts. The intersection of the two charts ${}_N\mathbb{R}_q^4$, ${}_S\mathbb{R}_q^4$ of $\mathcal{A}(S_q^4)$ admits a very natural geometric interpretation if we write the real, positive generator β^{-1} of ${}_{SN}\mathbb{R}_q^4$ as $\beta^{-1} = r^{-2}$. In Prop. 3.2 and Prop. 3.10 below, we show that by suitably rescaling the generators of ${}_{SN}\mathbb{R}_q^4$ by r^{-1} (or equivalently those of ${}_S\mathbb{R}_q^4$ by r) we can recover a classical $SU(2) \simeq S^3$ inside the intersection of the two charts.

Definition 3.1. We denote by $\widetilde{{}_{\text{SN}}\mathbb{R}_q^4}$ the $*$ -algebra generated by elements $\{\beta_{13}, \beta_{14}, \beta_{23}, \beta_{24}, r^{-1}\}$ with commutation relations

$$\beta_{13}r^{-1} = q^{-1}r^{-1}\beta_{13}, \quad \beta_{14}r^{-1} = q^{-1}r^{-1}\beta_{14}, \quad \beta_{23}r^{-1} = qr^{-1}\beta_{23}, \quad \beta_{24}r^{-1} = qr^{-1}\beta_{24}, \quad (3.1)$$

and satisfying

$$r^{-2}(\beta_{13}\beta_{24} - \beta_{14}\beta_{23}) = q = (\beta_{13}\beta_{24} - \beta_{14}\beta_{23})r^{-2}. \quad (3.2)$$

The $*$ -structure is the one given in (2.8), together with $(r^{-1})^* = r^{-1}$.

From (3.2) we can deduce that r^{-2} is invertible in $\widetilde{{}_{\text{SN}}\mathbb{R}_q^4}$ with inverse (denoted r^2) given by

$$r^2 := q^{-1}(\beta_{13}\beta_{24} - \beta_{14}\beta_{23}) = \beta_{24}^*\beta_{24} + \beta_{23}^*\beta_{23}. \quad (3.3)$$

Thus r^{-1} is invertible as well: denoting with r its inverse, the explicit expression is $r = r^2r^{-1} = r^{-1}r^2$. We note that r^2 is a combination of the β_{ij} 's alone, contrary to r which contains a contribution from r^{-1} as well.

The $*$ -algebras $\widetilde{{}_{\text{SN}}\mathbb{R}_q^4}$ and $\widetilde{{}_{\text{SN}}\mathbb{R}_q^4}$ are not isomorphic, nevertheless they describe the same 'quantum space'. Indeed at the C^* -algebra level the element $\alpha^{-1} = \beta_{24}^*\beta_{24} + \beta_{23}^*\beta_{23}$ is positive being sum of the two positive elements $\beta_{24}^*\beta_{24}$ and $\beta_{23}^*\beta_{23}$ (see e.g. [7, Thm. I.4.5, Cor. I.4.4]). Hence there exist by the general theory a (unique) element $r = r^* \in \widetilde{{}_{\text{SN}}\mathbb{R}_q^4}$ such that $\alpha^{-1} = r^2$, (see e.g. [7, Cor. I.4.1]).

Proposition 3.2. Denote \mathcal{A} the $*$ -subalgebra of $\widetilde{{}_{\text{SN}}\mathbb{R}_q^4}$ generated by the elements

$$x_{23} := \beta_{23}r^{-1}, \quad x_{24} := \beta_{24}r^{-1}, \quad x_{23}^* = r^{-1}\beta_{23}^*, \quad x_{24}^* = r^{-1}\beta_{24}^*. \quad (3.4)$$

Then in \mathcal{A} the sphere relation

$$x_{23}^*x_{23} + x_{24}^*x_{24} = 1$$

holds and \mathcal{A} is commutative.

Proof. The commutativity among the generators can be checked by a direct computation. The sphere relation is obtained by multiplying eq. (3.3) from both sides by r^{-1} . \square

For future convenience, let us organise the generators of the algebra \mathcal{A} as entries of a 2×2 matrix:

$$A := \begin{pmatrix} x_{23} & x_{24} \\ -x_{24}^* & x_{23}^* \end{pmatrix} \quad (3.5)$$

with $\det(A) = x_{23}x_{23}^* + x_{24}x_{24}^* = 1$.

Proposition 3.3. The $*$ -algebra \mathcal{A} can be provided with a Hopf algebra structure by introducing a coproduct, counit and antipode defined on the generators respectively via

$$\Delta(A) = A \otimes A, \quad \varepsilon(A) = \mathbb{I}, \quad S(A) = \begin{pmatrix} x_{23}^* & -x_{24} \\ x_{24}^* & x_{23} \end{pmatrix}.$$

The resulting Hopf algebra is the coordinate Hopf algebra $\mathcal{A}(\text{SU}(2))$ of $\text{SU}(2)$.

We now show that $\widetilde{{}_{SN}\mathbb{R}_q^4}$ can be factorized into a product of a commutative 3-sphere $S^3 \simeq SU(2)$ represented by \mathcal{A} and a 1-dimensional interval I , algebraically described as the $*$ -algebra \mathcal{I} generated by $\{r, r^{-1}\}$ satisfying the relation $rr^{-1} = r^{-1}r = 1$. While both \mathcal{A} and \mathcal{I} are commutative, the noncommutativity emerges from their tensor product. The factorization is expressed in Proposition 3.7 below. It is an elementary result from the algebraic point of view, nevertheless it provides a very nice geometrical picture.

Definition 3.4. We denote by $\mathcal{A} \otimes_{\Psi} \mathcal{I}$ the twisted tensor product algebra consisting of the vector space $\mathcal{A} \otimes \mathcal{I}$ endowed with the multiplication

$$m_{\theta} := (m_{\mathcal{A}} \otimes m_{\mathcal{I}})(\text{id}_{\mathcal{A}} \otimes \Psi \otimes \text{id}_{\mathcal{I}})$$

where $\Psi : \mathcal{I} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{I}$ is the linear map defined on the vector space base elements by

$$\Psi(r^{\pm n} \otimes x_{23}^a (x_{23}^*)^b x_{24}^c (x_{24}^*)^d) := q^{\pm n(a+c) \mp n(b+d)} x_{23}^a (x_{23}^*)^b x_{24}^c (x_{24}^*)^d \otimes r^{\pm n} \quad (3.6)$$

for all integers $n, a, b, c, d \in \mathbb{N} \cup \{0\}$.

We remark that the twist Ψ is normal: $\Psi(1 \otimes x) = x \otimes 1$ and $\Psi(r^{\pm a} \otimes 1) = 1 \otimes r^{\pm a}$, $\forall x \in \mathcal{A}$, $a \in \mathbb{N}$. Notice that Ψ is *not* an algebra morphism. On the algebra generators it reads

$$\begin{aligned} \Psi(r^{\pm 1} \otimes x_{23}) &= q^{\pm 1} x_{23} \otimes r^{\pm 1} & ; & \quad \Psi(r^{\pm 1} \otimes x_{23}^*) = q^{\mp 1} x_{23}^* \otimes r^{\pm 1} \\ \Psi(r^{\pm 1} \otimes x_{24}) &= q^{\pm 1} x_{24} \otimes r^{\pm 1} & ; & \quad \Psi(r^{\pm 1} \otimes x_{24}^*) = q^{\mp 1} x_{24}^* \otimes r^{\pm 1}. \end{aligned} \quad (3.7)$$

Lemma 3.5. The algebra $\mathcal{A} \otimes_{\Psi} \mathcal{I}$ is associative and unital.

Proof. From the general theory of twisted tensor product algebras, in order to prove that the multiplication m_{θ} is associative it is enough to prove that the normal twist Ψ satisfies the following two conditions (see e.g. [2]):

$$(\text{id}_{\mathcal{A}} \otimes m_{\mathcal{I}})(\Psi \otimes \text{id}_{\mathcal{I}})(\text{id}_{\mathcal{I}} \otimes \Psi) = \Psi(m_{\mathcal{I}} \otimes \text{id}_{\mathcal{A}}), \quad (3.8)$$

$$(m_{\mathcal{A}} \otimes \text{id}_{\mathcal{I}})(\text{id}_{\mathcal{A}} \otimes \Psi)(\Psi \otimes \text{id}_{\mathcal{A}}) = \Psi(\text{id}_{\mathcal{I}} \otimes m_{\mathcal{A}}) \quad (3.9)$$

which are easily proved by using the explicit form of the twist as given in (3.6). Finally, $1 \otimes 1$ is the unit in $\mathcal{A} \otimes_{\Psi} \mathcal{I}$ from the general theory of normal twists. \square

Lemma 3.6. The map $\Psi' = \Psi \circ \tau$ is $*$ -compatible. Then $\mathcal{A} \otimes_{\Psi} \mathcal{I}$ is a $*$ -algebra, with involution

$$*_{\Psi}(x \otimes j) := (x \otimes j)^{*_{\Psi}} := \Psi(j^* \otimes x^*), \forall x \in \mathcal{A}, j \in \mathcal{I}.$$

Proof. By direct check and as a direct application of Prop. A.2. \square

On the generators of $\mathcal{A} \otimes_{\Psi} \mathcal{I}$ the $*$ -structure reads

$$(x_{23} \otimes r^{\pm 1})^{*_{\Psi}} = q^{\mp 1} x_{23}^* \otimes r^{\pm 1} \quad ; \quad (x_{24} \otimes r^{\pm 1})^{*_{\Psi}} = q^{\mp 1} x_{24}^* \otimes r^{\pm 1}. \quad (3.10)$$

Proposition 3.7. The map $f_{SN} : \widetilde{{}_{SN}\mathbb{R}_q^4} \rightarrow \mathcal{A} \otimes_{\Psi} \mathcal{I}$ defined on the generators as

$$f_{SN}(\beta_{23}) = x_{23} \otimes r, \quad f_{SN}(\beta_{24}) = x_{24} \otimes r, \quad f_{SN}(r^{-1}) = 1 \otimes r^{-1} \quad (3.11)$$

and extended as a $*$ -algebra morphism, is an isomorphism of unital $*$ -algebras.

Proof. We have only to prove that the map f_{SN} preserves the commutation relations (3.1) among the generators and equation (3.2). We prove this latter, the other identities being proved in a similar way. We have

$$\begin{aligned} f_{SN}(r^{-2})f_{SN}(\beta_{24}^*\beta_{24} + \beta_{23}^*\beta_{23}) &= (1 \otimes r^{-2}) \left[q^{-1}(x_{24}^* \otimes r)(x_{24} \otimes r) + q^{-1}(x_{23}^* \otimes r)(x_{23} \otimes r) \right] \\ &= q \left[(x_{24}^* \otimes r^{-1})(x_{24} \otimes r) + (x_{23}^* \otimes r^{-1})(x_{23} \otimes r) \right] \\ &= (x_{23}^*x_{23} + x_{24}^*x_{24}) \otimes 1 = 1 \otimes 1. \end{aligned}$$

where we made use of (3.10). This also proves that f_{SN} preserves the units. \square

Remark 3.8. It is possible to get a commutative subalgebra in the intersection of the two charts also without rescaling by r^{-1} . Consider the subalgebra $\mathcal{A}' \subset {}_{NS}\mathbb{R}_q^4$ generated by

$$a_{13} := \alpha_{13}, \quad a_{14} := \alpha_{14}, \quad a_{23} := q^{-1}\beta\alpha_{23}, \quad a_{24} := q\alpha_{24}\beta. \quad (3.12)$$

We have the identity

$$a_{24}a_{13} - a_{23}a_{14} = 1 \quad (3.13)$$

and \mathcal{A}' is commutative. Therefore \mathcal{A}' can be endowed with a Hopf algebra structure as done for \mathcal{A} in Proposition 3.3, obtaining the coordinate Hopf algebra of $SL(2)$. But note that \mathcal{A}' is not a $*$ -subalgebra of ${}_{NS}\mathbb{R}_q^4$: from (2.12) we have

$$a_{23}^* = -q^{-4}a_{14}\beta, \quad a_{24}^* = q^{-2}\beta a_{13} \quad (3.14)$$

and the identity (3.13) becomes

$$q^2 a_{23}^* a_{23} + q^2 a_{24}^* a_{24} = \beta \quad (3.15)$$

hence describing the equation of a three-sphere with invertible but non central radius. To recover the coordinate Hopf algebra of $SU(2)$ we should introduce on \mathcal{A}' a different (i.e. not inherited from ${}_{NS}\mathbb{R}_q^4$) $*$ -structure, setting

$$\bar{a}_{24} := a_{13}, \quad \bar{a}_{23} := a_{14}.$$

We can of course obtain analogous results starting with the algebra ${}_{NS}\mathbb{R}_q^4$. We omit the proofs.

Definition 3.9. We denote by $\widetilde{{}_{NS}\mathbb{R}_q^4}$ the $*$ -algebra generated by $\{\alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, r\}$, together with commutation relations

$$\alpha_{13}r = q r \alpha_{13}, \quad \alpha_{14}r = q r \alpha_{14}, \quad \alpha_{23}r = q^{-1} r \alpha_{23}, \quad \alpha_{24}r = q^{-1} r \alpha_{24}, \quad (3.16)$$

and satisfying

$$r^2(\alpha_{24}\alpha_{13} - \alpha_{23}\alpha_{14}) = q = (\alpha_{24}\alpha_{13} - \alpha_{23}\alpha_{14})r^2. \quad (3.17)$$

The $*$ -structure is the one given in (2.12), together with $r^* = r$.

The inverse of r^2 in $\widetilde{{}_{NS}\mathbb{R}_q^4}$, denoted r^{-2} , is computed from (3.2):

$$r^{-2} := q^{-1}(\alpha_{24}\alpha_{13} - \alpha_{23}\alpha_{14}). \quad (3.18)$$

Also r is invertible, with $r^{-1} = r^{-2}r = rr^{-2}$. The map Q (see (2.13) and Corollary 2.5) induces a $*$ -algebra isomorphism $\bar{Q}: \widetilde{{}_{NS}\mathbb{R}_q^4} \rightarrow \widetilde{{}_{SN}\mathbb{R}_q^4}$ with $\bar{Q}(r^{-1}) = r$.

Proposition 3.10. *The subalgebra of $\widetilde{{}_{NS}\mathbb{R}_q^4}$ generated by*

$$y_{23} := q r \alpha_{23}, \quad y_{24} := q r \alpha_{24}, \quad y_{23}^* = q \alpha_{23}^* r, \quad y_{24}^* = q \alpha_{24}^* r \quad (3.19)$$

is commutative and coincides with $\widetilde{Q}^{-1}(\mathcal{A})$. Moreover, the following sphere relation holds:

$$y_{23}^* y_{23} + y_{24}^* y_{24} = 1.$$

Proposition 3.11. *The map $f_{NS} : \widetilde{{}_{NS}\mathbb{R}_q^4} \rightarrow \mathcal{A} \otimes_{\Psi} \mathcal{I}$ defined on the generators as*

$$f_{NS}(\alpha_{23}) = q^{-2} x_{23} \otimes r^{-1}, \quad f_{NS}(\alpha_{24}) = q^{-2} x_{24} \otimes r^{-1}, \quad f_{NS}(r) = 1 \otimes r \quad (3.20)$$

and extended as a $$ -algebra morphism, is an isomorphism of unital $*$ -algebras.*

We conclude by noting that from the above results, the diagram of $*$ -algebra isomorphisms below is commutative:

$$\begin{array}{ccc} \widetilde{{}_{NS}\mathbb{R}_q^4} & \xrightarrow{\widetilde{Q}} & \widetilde{{}_{SN}\mathbb{R}_q^4} \\ & \searrow f_{NS} & \swarrow f_{SN} \\ & \mathcal{A} \otimes_{\Psi} \mathcal{I} & \end{array} \quad (3.21)$$

3.2. The structure sheaf $\mathcal{O}_{S_q^4}$ and the differential calculus sheaf $\Omega_{S_q^4}^{\bullet}$. In classical geometry spaces can be characterized by their structure sheaf. More precisely, a topological space M together with a sheaf of commutative rings \mathcal{O}_M on M , referred to as the structure sheaf of M , form a *ringed space* (M, \mathcal{O}_M) . By specifying a local model for the sheaf \mathcal{O}_M we recover different geometrical notions. Consider for example the ringed spaces $(\mathbb{R}^n, \mathcal{C})$ and $(\mathbb{R}^n, \mathcal{C}^m)$, where \mathcal{C} is the sheaf of continuous functions on \mathbb{R}^n , $n \in \mathbb{N}$, and \mathcal{C}^m , $m \in \mathbb{N} \cup \{\infty\}$, is the sheaf of m -times differentiable functions on \mathbb{R}^n . Then if (M, \mathcal{O}_M) is locally isomorphic, as ringed space, to $(\mathbb{R}^n, \mathcal{C})$ (resp. $(\mathbb{R}^n, \mathcal{C}^m)$) we say that M is a topological manifold (resp. differentiable m -manifold) of dimension n . Further different choices of the local model characterize analytical manifolds, complex manifolds, schemes [12, Ex. 2.4]. Following [12], we call a *quantum space* over M a ringed space (M, \mathcal{O}_M) where \mathcal{O}_M is now a sheaf of not necessarily commutative algebras. In this framework the 'local charts' of the noncommutative algebra $\mathcal{A}(S_q^4)$ naturally define a quantum space over the classical 4-sphere S^4 , as we are going to show.

Let us consider the topology of S^4 whose basis consists of the following sets: $U_N := S^4 \setminus \{NP\}$, $U_S := S^4 \setminus \{SP\}$ and their intersection $U_{SN} := S^4 \setminus \{NP, SP\}$. Here NP , SP denote the North and South poles respectively. We construct a sheaf $\mathcal{O}_{S_q^4}$ of noncommutative $*$ -algebras on S^4 by the assignment

$$\mathcal{O}_{S_q^4}(U_N) := {}_N\mathbb{R}_q^4, \quad \mathcal{O}_{S_q^4}(U_S) := {}_S\mathbb{R}_q^4, \quad \mathcal{O}_{S_q^4}(U_{SN}) := \widetilde{{}_{SN}\mathbb{R}_q^4} \simeq \mathcal{A} \otimes_{\Psi} \mathcal{I} \quad (3.22)$$

together with restriction maps

$$\begin{aligned} \rho_{N,SN} &: {}_N\mathbb{R}_q^4 \rightarrow \widetilde{{}_{SN}\mathbb{R}_q^4}, & \beta_{ij} &\mapsto \beta_{ij} \\ \rho_{S,SN} &: {}_S\mathbb{R}_q^4 \rightarrow \widetilde{{}_{SN}\mathbb{R}_q^4}, & \alpha_{ij} &\mapsto \widetilde{Q}(\alpha_{ij}) \end{aligned} \quad (3.23)$$

and identities otherwise. We observe that $\mathcal{O}_{S_q^4}$ is not a flabby sheaf, i.e. its restriction maps are not surjective. By standard arguments in sheaf theory, $\mathcal{O}_{S_q^4}$ is defined on the whole topology of S^4 , with in particular

$$\begin{aligned} \mathcal{O}_{S_q^4}(S^4) &= \mathcal{O}_{S_q^4}(U_N \cup U_S) \\ &= \{(a_N, a_S, a_{SN}) \in \mathcal{O}_{S_q^4}(U_N) \oplus \mathcal{O}_{S_q^4}(U_S) \oplus \mathcal{O}_{S_q^4}(U_{SN}) \mid \rho_{N,SN}(a_N) = \rho_{S,SN}(a_S) = a_{SN}\} \\ &\simeq \{(a_N, a_S) \in \mathcal{O}_{S_q^4}(U_N) \oplus \mathcal{O}_{S_q^4}(U_S) \mid \rho_{N,SN}(a_N) = \rho_{S,SN}(a_S)\} \end{aligned} \quad (3.24)$$

Definition 3.12. *The quantum space of the noncommutative 4-sphere S_q^4 is the sheaf of noncommutative algebras $\mathcal{O}_{S_q^4}$ over the classical 4-sphere S^4 .*

Remark 3.13. *The isomorphism \mathcal{Q} of Proposition 2.4 ensures that the quantum space S_q^4 has as local model the same quantum deformation $\mathbb{R}_q^4 := {}_N\mathbb{R}_q^4 \simeq {}_S\mathbb{R}_q^4$ of \mathbb{R}^4 , i.e. the quantum sphere is locally isomorphic to (isomorphic copies of) \mathbb{R}_q^4 . Quantum spaces generally lack a unique local model, in the sense that they may host non-isomorphic noncommutative deformations in different patches. For example the twistor bundle over S_q^4 has local isomorphisms to $\mathbb{R}_q^4 \otimes_q \mathbb{CP}_q^1$ and to $\mathbb{R}_q^4 \otimes \mathbb{CP}^1$ depending on the two patches [4, §4.6].*

Also differential forms admit a natural description in terms of sheaves, and so do their noncommutative deformations. The noncommutative differential calculus of S_q^4 was written on the local patches ${}_N\mathbb{R}_q^4$ and ${}_S\mathbb{R}_q^4$, as well as on their intersection, in [4, §6.2]. Here we describe it via the data of a sheaf $\Omega_{S_q^4}^\bullet$ of graded algebras.

Let us start with the open set $U_N \subset S^4$. The structure sheaf reads $\mathcal{O}_{S_q^4}(U_N) = {}_N\mathbb{R}_q^4$, and the algebra ${}_N\mathbb{R}_q^4$ admits a canonical noncommutative differential $*$ -calculus ${}_N\Omega_q^\bullet = \wedge_q^\bullet {}_N\Omega_q^1$. The latter is a differential graded $*$ -algebra whose degree zero part coincides with ${}_N\mathbb{R}_q^4$ and whose degree one part has generators $\{d\beta_{23}, (d\beta_{23})^*, d\beta_{24}, (d\beta_{24})^*\}$. The differential $d : {}_N\mathbb{R}_q^4 \rightarrow {}_N\Omega_q^1$ is defined as

$$d(\beta_{23}) = d\beta_{23}, \quad d(\beta_{23}^*) = (d\beta_{23})^*, \quad d(\beta_{24}) = d\beta_{24}, \quad d(\beta_{24}^*) = (d\beta_{24})^*$$

and extended uniquely to a degree one $*$ -map $d : {}_N\Omega_q^m \rightarrow {}_N\Omega_q^{m+1}$ via \mathbb{C} -linearity, the requirement $d^2 = 0$, and the Leibniz rule

$$d(\omega \wedge_q \omega') = d\omega \wedge_q \omega' + (-1)^{|\omega|} \omega \wedge_q d\omega'. \quad (3.25)$$

Since there is no risk of confusion, we will simply write $d\beta_{23}^*$, resp. $d\beta_{24}^*$ to indicate $(d\beta_{23})^*$, resp. $(d\beta_{24})^*$. The commutation relations in ${}_N\Omega_q^\bullet$ are completely derived from the ones involving elements of degree zero and one. To simplify the notation we drop the symbol \wedge_q in formulas; in addition to the relations in ${}_N\mathbb{R}_q^4$ we have

$$\begin{aligned} \beta_{23}d\beta_{23} &= d\beta_{23}\beta_{23}, \quad \beta_{23}(d\beta_{23})^* = q^{-2}(d\beta_{23})^*\beta_{23}, \quad \beta_{23}d\beta_{24} = d\beta_{24}\beta_{23}, \quad \beta_{23}(d\beta_{24})^* = q^{-2}(d\beta_{24})^*\beta_{23}, \\ \beta_{24}d\beta_{24} &= d\beta_{24}\beta_{24}, \quad \beta_{24}(d\beta_{24})^* = q^{-2}(d\beta_{24})^*\beta_{24}, \quad \beta_{24}d\beta_{23} = d\beta_{23}\beta_{24}, \quad \beta_{24}(d\beta_{23})^* = q^{-2}(d\beta_{23})^*\beta_{24}. \end{aligned}$$

By applying the maps $*$ and d one gets the remaining relations in degree one, and from there the ones in higher degrees.

A completely analogous construction is carried out in the second chart $U_S \subset S^4$. The algebra ${}_S\mathbb{R}_q^4$ admits a canonical noncommutative differential calculus ${}_S\Omega_q^\bullet = \wedge_q^\bullet {}_S\Omega_q^1$, built out of ${}_S\Omega_q^0 := {}_S\mathbb{R}_q^4$ and the differential $d : {}_S\mathbb{R}_q^4 \rightarrow {}_S\Omega_q^1$, $d(\alpha_{ij}) \mapsto d\alpha_{ij}$, following the same prescriptions as above. Or, even more directly, one can extend the $*$ -isomorphism $Q : {}_S\mathbb{R}_q^4 \rightarrow {}_N\mathbb{R}_q^4$ to 1-forms, putting $q^2 d\alpha_{23} = Q^{-1}(d\beta_{23}) := dQ^{-1}(\beta_{23})$ and so on (the q -coefficients come from the definition of Q in (2.13)), hence realizing the commutative diagram

$$\begin{array}{ccc} {}_S\mathbb{R}_q^4 & \xrightarrow{Q} & {}_N\mathbb{R}_q^4 \\ d \downarrow & & \downarrow d \\ {}_S\Omega_q^1 & \xrightarrow{Q} & {}_N\Omega_q^1 \end{array} \quad (3.26)$$

where ${}_S\Omega_q^1 := Q^{-1}({}_N\Omega_q^1)$; then the isomorphism extends to the whole differential calculus, ${}_S\Omega_q^\bullet = Q^{-1}({}_N\Omega_q^\bullet)$.

In the intersection U_{SN} we consider the noncommutative differential calculus ${}_{SN}\Omega_q^\bullet$ of the algebra $\widetilde{{}_{SN}\mathbb{R}_q^4}$. Again this can be done following the standard construction, or more directly taking advantage of the commutative diagram

$$\begin{array}{ccc} {}_S\mathbb{R}_q^4 & \longrightarrow & \widetilde{{}_{SN}\mathbb{R}_q^4} \\ d \downarrow & & \downarrow d \\ {}_S\Omega_q^\bullet & \longrightarrow & {}_{SN}\Omega_q^\bullet \end{array} \quad (3.27)$$

which emphasizes how the process of algebraic extension commutes with the differential. In this way ${}_{SN}\Omega_q^1$ is obtained as the extension of ${}_N\Omega_q^1$ by an extra-generator $dr^{-1} = (dr^{-1})^*$; the additional commutation relations involving dr^{-1} are

$$\begin{aligned} r dr^{-1} &= dr^{-1} r & \beta_{23} dr^{-1} &= q^2 dr^{-1} \beta_{23}, & \beta_{24} dr^{-1} &= q^2 dr^{-1} \beta_{24}, \\ \beta_{23}^* dr^{-1} &= q^{-2} dr^{-1} d\beta_{23}^*, & \beta_{24}^* dr^{-1} &= q^{-2} dr^{-1} d\beta_{24}^*, \end{aligned}$$

together with those obtained from them by applying the maps d and $*$. Finally, we have the differential counterpart of the sphere relation $r^{-2}(\beta_{23}^* \beta_{23} + \beta_{24}^* \beta_{24}) = 1$ (see (3.3)), which reads

$$r^{-1} dr^{-1} + (dr^{-1}) r^{-1} + (d\beta_{23})^* \beta_{23} + \beta_{23}^* d\beta_{23} + (d\beta_{24})^* \beta_{24} + \beta_{24}^* d\beta_{24} = 0. \quad (3.28)$$

These noncommutative differential calculi are arranged into a sheaf $\Omega_{S_q^4}^\bullet$ on S^4 by the assignment

$$\Omega_{S_q^4}^\bullet(U_N) := {}_N\Omega_q^\bullet, \quad \Omega_{S_q^4}^\bullet(U_S) := {}_S\Omega_q^\bullet, \quad \Omega_{S_q^4}^\bullet(U_{SN}) := {}_{SN}\Omega_q^\bullet \quad (3.29)$$

together with restriction maps

$$\begin{aligned} \rho_{N,SN} : {}_N\Omega_q^\bullet &\rightarrow {}_{SN}\Omega_q^\bullet, & d\beta_{ij} &\mapsto d\beta_{ij} \\ \rho_{S,SN} : {}_S\Omega_q^\bullet &\rightarrow {}_{SN}\Omega_q^\bullet, & d\alpha_{ij} &\mapsto \widetilde{Q}(d\alpha_{ij}) = d(\widetilde{Q}(\alpha_{ij})) \end{aligned}$$

Definition 3.14. *The noncommutative differential calculus of the noncommutative 4-sphere S_q^4 is the sheaf of noncommutative algebras $\Omega_{S_q^4}^\bullet$ over the classical 4-sphere S^4 .*

There is a Hodge duality operator on $\Omega_{S_q^4}^2$, which will be important when discussing the self-duality of the instanton connection. As discussed in more detail in [4, §5.3], it coincides with the classical Hodge operator on $\Omega_{S^4}^\bullet$ since the toric symmetry beyond the 2-cocycle deformation induces a conformal transformation of the classical metric on S^4 .

We start on the local chart U_N , making use of the classical Hodge duality in \mathbb{R}^4 . Denote by x_i ($i = 0, 1, 2, 3$) the coordinate functions on \mathbb{R}^4 . The Hodge duality operator on two forms $\star : \Omega^2(\mathbb{R}^4) \rightarrow \Omega^2(\mathbb{R}^4)$ squares to the identity, and provides a decomposition of $\Omega^2(\mathbb{R}^4)$ into self-dual (eigenvalue +1) and anti-selfdual (eigenvalue -1) differential forms,

$$\Omega^2(\mathbb{R}^4) = \Omega^{2,+}(\mathbb{R}^4) \oplus \Omega^{2,-}(\mathbb{R}^4).$$

As $\mathcal{A}(\mathbb{R}^4)$ -left modules the summands are generated by:

$$\Omega^{2,+}(\mathbb{R}^4) = \langle l_1^+ = dx_0 dx_1 + dx_2 dx_3, l_2^+ = dx_0 dx_2 - dx_1 dx_3, l_3^+ = dx_0 dx_3 + dx_1 dx_2 \rangle \quad (3.30)$$

$$\Omega^{2,-}(\mathbb{R}^4) = \langle l_1^- = dx_0 dx_1 - dx_2 dx_3, l_2^- = dx_0 dx_2 + dx_1 dx_3, l_3^- = dx_0 dx_3 - dx_1 dx_2 \rangle$$

To match our previous notation with ${}_N\mathbb{R}_q^4$ and its generators, it is more convenient to consider complex coordinates

$$2\beta_{23} = x_0 + ix_1, \quad 2\beta_{23}^* = x_0 - ix_1, \quad 2\beta_{24} = x_2 + ix_3, \quad 2\beta_{24}^* = x_2 - ix_3 \quad (3.31)$$

and, in view of our computations on the instanton curvature later on, to generate $\Omega^{2,-}(\mathbb{R}^4)$ with:

$$l_1^- = d\beta_{23}^* d\beta_{23} - d\beta_{24}^* d\beta_{24}, \quad l_2^- + l_3^- = d\beta_{23}^* d\beta_{24}, \quad l_2^- - l_3^- = d\beta_{24}^* d\beta_{23}. \quad (3.32)$$

When now the β generators satisfy commutation relation (2.6) in ${}_N\mathbb{R}_q^4$ (using (2.8) as well) the above expressions define anti-selfdual two-forms in ${}_N\Omega_q^\bullet$. A completely analogous construction is performed in the chart U_S . By using the restriction maps of Ω_q^\bullet these local assignments define the sheaf (of $\mathcal{O}_{S_q^4}$ -modules) of anti-selfdual 2-forms $\Omega_q^{2,-}$.

4. THE QUANTUM HOPF BUNDLE AS A SHEAF OF HOPF-GALOIS EXTENSIONS

We now introduce a quantum principal bundle \mathcal{P} - in the sense of [12, §3] - with structure group $\mathcal{A} = \mathcal{A}(SU(2))$ on the quantum 4-sphere S_q^4 . We construct it out of a family of linear maps τ_{ij} , playing the role of ‘transition functions’ from the structure group \mathcal{A} to the double intersections of the two charts $\mathcal{O}_{S_q^4}(U_N) = {}_N\mathbb{R}_q^4$ and $\mathcal{O}_{S_q^4}(U_S) = {}_S\mathbb{R}_q^4$.

We start by recalling some results on Hopf comodule algebras and Hopf-Galois extensions. Let H be a cosemisimple Hopf algebra with bijective antipode; if not otherwise stated, from now on we will always work with Hopf algebras of this kind. Let P be a right H -comodule algebra, namely we have a right H -coaction $\delta_H : P \rightarrow P \otimes H$, $\delta_H(p) = p_{(0)} \otimes p_{(1)}$, which is an algebra morphism. The space of coinvariants $P^{coH} := \{p \in P \text{ s.t. } \delta_H(p) = p \otimes 1\}$ is a subalgebra of P . The map $\chi : P \otimes_{P^{coH}} P \rightarrow P \otimes H$ defined as $p' \otimes p \mapsto p' p_{(0)} \otimes p_{(1)}$ is usually referred to as the canonical map. When χ is bijective we say that $P^{coH} \subset P$ is an H -Hopf-Galois extension. An H -(Hopf-Galois) extension $B \subset P$ is cleft if and only if it is isomorphic to a crossed product $P \simeq B \sharp_{\sigma} H$ [11, Thm. 7.2.2]. A morphism of H -comodule algebras

$\phi : P \rightarrow P'$ is an algebra morphism which intertwines the H -coactions: $(\phi \otimes \text{id}_H)\delta_H = \delta'_H\phi$. The following properties are easily verified:

- (a) ϕ maps coinvariants into coinvariants, $\phi(P^{\text{co}H}) \subset P'^{\text{co}H}$;
- (b) $\phi \otimes \phi$ is well defined on $P \otimes_{P^{\text{co}H}} P$ and $(\phi \otimes \phi)(P \otimes_{P^{\text{co}H}} P) \subset P' \otimes_{P'^{\text{co}H}} P'$;
- (c) ϕ intertwines the canonical maps, in the sense that $\chi'(\phi \otimes \phi) = (\phi \otimes \text{id}_H)\chi$ on $P \otimes_{P^{\text{co}H}} P$, and their lifts $\tilde{\chi} : P \otimes P \rightarrow P \otimes H$ and $\tilde{\chi}' : P' \otimes P' \rightarrow P' \otimes H$.

Finally, we recall that an extension $P^{\text{co}H} \subset P$ is Hopf-Galois if and only if it admits a strong connection (see e.g. [8, §2.4] and reference therein). The latter is a unital linear map $l : H \rightarrow P \otimes P$, $l(h) = l(h)^{<1>} \otimes l(h)^{<2>}$, satisfying:

- (i) $(l \otimes \text{id}_H)\Delta_H = (\text{id}_P \otimes \delta_H)l$
- (ii) $(\text{id}_H \otimes l)\Delta_H = ({}_H\delta \otimes \text{id}_P)l$
- (iii) $\tilde{\chi} \circ l = 1_P \otimes \text{id}_H$

where ${}_H\delta : P \rightarrow H \otimes P$, ${}_H\delta(p) = S^{-1}(p_{(1)}) \otimes p_{(0)}$, is the left H -coaction induced from the right one δ_H . Given a strong connection l , the inverse of the canonical map χ is written as $\chi^{-1}(p \otimes h) = pl(h)^{<1>} \otimes_{P^{\text{co}H}} l(h)^{<2>}$.

Lemma 4.1. *Let $\phi : P \rightarrow P'$ be a morphism of unital right H -comodule algebras. If P is an H -Hopf-Galois extension, so does P' .*

Proof. By hypothesis we have a strong connection $l : H \rightarrow P \otimes P$. We claim that $l' := (\phi \otimes \phi)l : H \rightarrow P' \otimes P'$ is a strong connection on P' . Property (i) on l' amounts to the commutativity of the diagram

$$\begin{array}{ccccc} H & \xrightarrow{l} & P \otimes P & \xrightarrow{\phi \otimes \phi} & P' \otimes P' \\ \Delta_H \downarrow & & \downarrow \text{id}_P \otimes \delta_H & & \downarrow \text{id}_{P'} \otimes \delta'_H \\ H \otimes H & \xrightarrow{l \otimes \text{id}_H} & P \otimes P \otimes H & \xrightarrow{\phi \otimes \phi \otimes \text{id}_H} & P' \otimes P' \otimes H \end{array} \quad (4.1)$$

which follows from the commutativity of the two sub-diagrams (resp. by property (i) on l and since ϕ is an H -comodule algebra morphism). Similarly for property (ii). Finally property (iii) on l' is displayed as the commutativity of the diagram

$$\begin{array}{ccccc} H & \xrightarrow{l} & P \otimes P & \xrightarrow{\phi \otimes \phi} & P' \otimes P' \\ & \searrow 1_P \otimes \text{id}_H & \downarrow \tilde{\chi} & & \downarrow \tilde{\chi}' \\ & & P \otimes H & \xrightarrow{\phi \otimes \text{id}_H} & P' \otimes H \end{array} \quad (4.2)$$

which is again a consequence of the commutativity of the two sub-diagrams. \square

Remark 4.2. *The geometric counterpart of the previous Lemma is the well known fact that given two G -spaces X and Y and a G -equivariant morphism $f : X \rightarrow Y$, if the G -action on Y is free so does the one on X .*

Definition 4.3. *Let X be a topological space. Let \mathcal{F} be a sheaf of (not necessarily commutative) algebras over X and H a Hopf algebra. We say that \mathcal{F} is a sheaf of H -Hopf-Galois extensions if:*

- (i) \mathcal{F} is a sheaf of (say) right H -comodule algebras and for each $W \subset U$ the restriction map $\rho_{UW} : \mathcal{F}(U) \rightarrow \mathcal{F}(W)$ is a morphism of H -comodule algebras;

(ii) for each $U \subset X$ open set, $\mathcal{F}(U)^{\text{co}(H)} \subseteq \mathcal{F}(U)$ is a Hopf-Galois extension.

We denote by $\mathcal{F}^{\text{co}(H)}$ the sheaf on X which associate to each open set U the subalgebra of coinvariants $\mathcal{F}(U)^{\text{co}(H)}$; we call it the subsheaf of coinvariants.

Note that by Lemma 4.1 applied to the restriction maps, once $\mathcal{F}(U)$ is an H -Hopf-Galois extension, then $\mathcal{F}(W)$ is an H -Hopf-Galois extension for any open set $W \subseteq U$. In particular, if the algebra of global sections $\mathcal{F}(X)$ is an H -Hopf-Galois extension then automatically \mathcal{F} is a sheaf of H -Hopf-Galois extensions over X . We then see that the property of being a Hopf-Galois extension restricts locally. The converse ‘gluing property’ is true for flabby sheaves (i.e. when restriction maps are surjective), from the general theory of piecewise principality, see [8, Thm. 3.3 and Corol. 3.10]. Namely, given a flabby sheaf \mathcal{F} and an open set U with a covering $\{U_i\}_{i \in I}$, if $\mathcal{F}(U_i)$ is an H -Hopf-Galois extension for any $i \in I$ then also $\mathcal{F}(U)$ is an H -Hopf-Galois extension². A natural class of examples of sheaves of Hopf-Galois extensions comes from smooth principal bundles.

Example 4.4. Let M be a smooth manifold, G a matrix Lie group and $\pi : P \rightarrow M$ a smooth principal G -bundle over M which locally trivializes with respect to some covering $\{U_i\}_{i \in I}$ of M , namely $\pi^{-1}(U_i) \simeq U_i \times G$. Let H be the Hopf algebra of coordinate functions on G , and \mathcal{M} the sheaf of smooth functions on M . We can use the local trivialization of P to define a sheaf \mathcal{F} of H -Hopf-Galois extensions over M as follows: for each U_i set $\mathcal{F}(U_i) = \mathcal{M}(U_i) \otimes H$. The restriction maps are the restriction maps of \mathcal{M} tensored with the identity on H , so they are surjective. Each $\mathcal{F}(U_i)$ is easily seen to be an H -Hopf-Galois extension (in particular cleft), and by the above discussed gluing property also $\mathcal{F}(M)$, which geometrically corresponds to the algebra of smooth functions on P , is an H -Hopf-Galois extension (albeit in general not a cleft one).

Example 4.5. From the same smooth principal G -bundle $\pi : P \rightarrow M$ one can also define the sheaf \mathcal{P} of smooth functions on P . The free and proper G -action on P translates in $\mathcal{P}(P)$, the global sections, being an H -Hopf-Galois extension. By Lemma 4.1 each local restriction $\mathcal{P}(V)$, for $V \subset P$ open, is automatically an H -Hopf-Galois extension. In particular one can consider the covering of P given by $\{\pi^{-1}(U_i)\}_{i \in I}$, where $\{U_i\}_{i \in I}$ is a covering of M which locally trivializes the principal bundle; in this case the local restrictions turn out to be cleft H -Hopf algebra extensions, $\mathcal{P}(\pi^{-1}(U_i)) = \mathcal{M}(U_i) \otimes H$. Note that on open sets of the form $\pi^{-1}(U) \subset P$, for $U \subset M$, the sheaf \mathcal{P} agrees with the sheaf \mathcal{F} over M of Example 4.4, $\mathcal{P}(\pi^{-1}(U)) = \mathcal{F}(U)$.

The previous examples naturally suggests the following definition.

Definition 4.6. A sheaf \mathcal{F} of H -Hopf-Galois extensions over a topological space X is called locally cleft if there exists an open covering $\{U_i\}_{i \in I}$ of X such that $\mathcal{F}(U_i)$ is cleft, $\forall i \in I$.

The notions of quantum principal bundle introduced by Pflaum and that of locally cleft sheaf of Hopf-Galois extensions are closely related.

Proposition 4.7. A sufficient condition for a quantum principal bundle \mathcal{P} to be a sheaf of Hopf-Galois extensions (in fact, locally cleft) is that \mathcal{P} is a flabby sheaf. In the opposite direction, every locally cleft sheaf of Hopf-Galois extensions is a quantum principal bundle.

²In [8] the authors consider principal extensions, which for H cosemisimple are equivalent to Hopf-Galois extensions, and they work with a family of H -comodule surjections $\pi_i : P \rightarrow P_i$, $i \in I$, such that $\cap_i (\ker \pi_i) = 0$. In Theorem 3.3 they show that P is principal if and only if the P_i 's are principal. To recover our setting it suffices to let π_i be the restriction map $\mathcal{F}(X) \rightarrow \mathcal{F}(U_i)$, as they point out in Corollary 3.10.

Proof. Let us consider a quantum principal bundle \mathcal{P} (with base quantum space X) on X . According to [12, Def. 3.1], there exists an open covering $\{U_i\}_{i \in I}$ of X and a family of sheaf isomorphisms $\Omega_i : \mathcal{X}(U_i) \#_i H \rightarrow \mathcal{P}(U_i)$, hence \mathcal{P} restricts locally to cleft Hopf-Galois extensions. For these to glue to a well defined Hopf-Galois sheaf, we have already discussed that a sufficient condition is that the restriction maps are surjective [8, Corol. 3.10].

For the second statement, suppose that \mathcal{P} is a locally cleft sheaf, say over X , of Hopf-Galois extensions. Then we have an injective sheaf morphism $\rho : \mathcal{P}^{co(H)} \hookrightarrow \mathcal{P}$ and an open covering $\{U_i\}_{i \in I}$ of X such that $\mathcal{P}(U_i)$ is cleft. The H -comodule sheaf morphisms $\Omega_i(U) : \mathcal{P}^{co(H)}(U) \#_i H \rightarrow \mathcal{P}(U)$, defined as $f \otimes h \mapsto f \gamma_i(h)$, where $U \subset U_i$ and $\gamma_i : H \rightarrow \mathcal{P}(U_i)$ is the cleaving map, are in fact isomorphism [11, Thm. 7.2.2]. It is now trivial to check that the datum of $(\mathcal{P}, \mathcal{P}^{co(H)}, \rho, H, (\Omega_i)_{i \in I})$ corresponds to Pflaum's definition of quantum principal bundle. \square

In the remaining of the section we construct a quantum principal bundle \mathcal{P} over the noncommutative 4-sphere S_q^4 . It mimics the sheaf of the principal $SU(2)$ -Hopf bundle, therefore it will be referred to as the quantum Hopf bundle over S_q^4 . We will see that it is not flabby, nevertheless we prove it to be a sheaf of locally cleft Hopf-Galois extensions. This is an example of how the flabbiness hypothesis in Proposition 4.7 is sufficient but not necessary.

We consider the covering of S^4 consisting of the two open sets U_N, U_S as before and the sheaf $\mathcal{O}_{S_q^4}$ introduced in (3.22) above. By using Prop. 3.7 (and understanding the isomorphism $\mathcal{A} \otimes_{\Psi} \mathcal{I} \simeq \mathcal{O}_{S_q^4}(U_N \cap U_S)$) we introduce 'transition functions' τ_{ij} , $i, j \in \{N, S\}$ as the linear maps

$$\begin{aligned} \tau_{NN} : \mathcal{A} &\rightarrow \mathcal{A} \otimes_{\Psi} \mathcal{I}, & \tau_{SS} : \mathcal{A} &\rightarrow \mathcal{A} \otimes_{\Psi} \mathcal{I}, & \tau_{NS} : \mathcal{A} &\rightarrow \mathcal{A} \otimes_{\Psi} \mathcal{I}, & \tau_{SN} : \mathcal{A} &\rightarrow \mathcal{A} \otimes_{\Psi} \mathcal{I} \\ h &\mapsto \varepsilon(h)1 \otimes 1, & h &\mapsto \varepsilon(h)1 \otimes 1, & h &\mapsto h \otimes 1, & h &\mapsto S(h) \otimes 1 \end{aligned} \quad (4.3)$$

for each $h \in \mathcal{A}$. It is promptly proved that the maps τ_{ij} form an \mathcal{A} -cocycle in $\mathcal{O}_{S_q^4}$ in the sense of [12, Def. 3.11]. Moreover the τ_{ij} , $i, j \in \{N, S\}$ above are algebra morphisms (τ_{SN} as well, despite the presence of the antipode, since \mathcal{A} is commutative).

The general theory gives a receipt for constructing a quantum principal bundle out of a set of transition functions:

$$\begin{aligned} \mathcal{P}(U_N) &:= \left\{ (b^N, b^{SN}) \in \left(\mathcal{O}_{S_q^4}(U_N) \otimes \mathcal{A} \right) \oplus \left(\mathcal{O}_{S_q^4}(U_S \cap U_N) \otimes \mathcal{A} \right) \right. \\ &\quad \left. \text{s.t. } (\rho_{N,SN} \otimes id)(b^N) = (m \otimes id)(id \otimes f_{SN}^{-1} \circ \tau_{NS} \otimes id)(id \otimes \Delta)(b^{SN}) \right\} \end{aligned} \quad (4.4)$$

similarly,

$$\begin{aligned} \mathcal{P}(U_S) &:= \left\{ (b^S, b^{SN}) \in \left(\mathcal{O}_{S_q^4}(U_S) \otimes \mathcal{A} \right) \oplus \left(\mathcal{O}_{S_q^4}(U_S \cap U_N) \otimes \mathcal{A} \right) \right. \\ &\quad \left. \text{s.t. } (\rho_{S,SN} \otimes id)(b^S) = (m \otimes id)(id \otimes f_{SN}^{-1} \circ \tau_{SN} \otimes id)(id \otimes \Delta)(b^{SN}) \right\} \end{aligned} \quad (4.5)$$

while on the intersection \mathcal{P} is simply given by

$$\mathcal{P}(U_S \cap U_N) := \mathcal{O}_{S_q^4}(U_S \cap U_N) \otimes \mathcal{A}. \quad (4.6)$$

Finally

$$\begin{aligned} \mathcal{P}(S^4) &:= \{(b^N, b^S) \in (\mathcal{O}_{S_q^4}(U_N) \otimes \mathcal{A}) \oplus (\mathcal{O}_{S_q^4}(U_S) \otimes \mathcal{A}) \\ &\quad \text{s.t. } (\rho_{N,SN} \otimes id)(b^N) = (m \otimes id)(id \otimes f_{SN}^{-1} \circ \tau_{NS} \otimes id)(id \otimes \Delta)(\rho_{S,SN} \otimes id)(b^S)\} . \end{aligned} \quad (4.7)$$

We note that we can constructively find pairs (b^N, b^{SN}) belonging to $\mathcal{P}(U_N)$ by defining (see [12, Lemma 3.13])

$$b^{SN} := (m \otimes id)(id \otimes f_{SN}^{-1} \circ \tau_{NS} \otimes id)(id \otimes \Delta)(\rho_{N,SN} \otimes id)b^N .$$

Hence one defines the trivialization morphism

$$\begin{aligned} \Omega_N : \mathcal{O}_{S_q^4}(U_N) \otimes \mathcal{A} &\rightarrow \mathcal{P}(U_N) \\ b^N &\mapsto (b^N, (m \otimes id)(id \otimes f_{SN}^{-1} \circ \tau_{NS} \otimes id)(id \otimes \Delta)(\rho_{N,SN} \otimes id)b^N) \end{aligned} \quad (4.8)$$

which is an isomorphism of right \mathcal{A} -comodule algebras. The same construction applies to pairs in $\mathcal{P}(U_S)$ and to the trivialization morphism $\Omega_S : \mathcal{O}_{S_q^4}(U_S) \otimes \mathcal{A} \rightarrow \mathcal{P}(U_S)$.

We already pointed out that the restriction maps $\rho_{N,SN}$ and $\rho_{S,SN}$ of the sheaf $\mathcal{O}_{S_q^4}$ are not surjective. As a consequence, also the sheaf \mathcal{P} of the quantum Hopf bundle is not flabby. Let us analyze the global sections of the sheaf \mathcal{P} . Clearly, the right regular corepresentation of \mathcal{A} on itself induces right coactions on the two summand of the algebra

$$\mathcal{P}_0(S^4) = ({}_N\mathbb{R}_q^4 \otimes \mathcal{A}) \oplus ({}_S\mathbb{R}_q^4 \otimes \mathcal{A}) .$$

Setting $B = {}_N\mathbb{R}_q^4$ or ${}_S\mathbb{R}_q^4$, the algebra $Y := B \otimes \mathcal{A}$ is a right \mathcal{A} -comodule algebra via $id \otimes \Delta$. Furthermore the corresponding canonical map (left Y -linear)

$$\chi : Y \otimes_{Y^{co}} Y \rightarrow Y \otimes \mathcal{A}, \quad (1 \otimes 1) \otimes (b \otimes g) \mapsto (b \otimes g_{(1)}) \otimes g_{(2)}$$

is invertible, with inverse given in terms of the antipode as

$$\chi^{-1} : (1 \otimes 1) \otimes h \mapsto (1 \otimes S(h_{(1)})) \otimes (1 \otimes h_{(2)}) . \quad (4.9)$$

(To prove that $\chi^{-1}\chi = id$ use that $Im(\chi^{-1}) \subset Y \otimes_{Y^{co}} Y$). In other words $B \simeq Y^{co} \subseteq Y$ is a Hopf-Galois extension, in particular a cleft one. Furthermore, since \mathcal{A} is cosemisimple, Y is faithfully flat as left B -module. Then

$$\begin{aligned} \mathcal{P}_0(S^4) &\longrightarrow ({}_N\mathbb{R}_q^4 \otimes \mathcal{A}^{\otimes 2}) \oplus ({}_S\mathbb{R}_q^4 \otimes \mathcal{A}^{\otimes 2}) \simeq \mathcal{P}_0(S^4) \otimes \mathcal{A} \\ (x \otimes g, y \otimes h) &\mapsto (x \otimes g_{(1)}, 0) \otimes g_{(2)} + (0, y \otimes h_{(1)}) \otimes h_{(2)} \end{aligned} \quad (4.10)$$

makes $\mathcal{P}_0(S^4)$ into an \mathcal{A} -comodule algebra.

The sheaf \mathcal{P} on S^4 is defined (cf. (4.7) above) as the pullback of the maps $\rho_{N,SN} \otimes id$ and $(m \otimes id)(id \otimes \tau_{NS} \otimes id)(id \otimes \Delta)(\rho_{S,SN} \otimes id)$. By construction, \mathcal{P} is a sheaf of right \mathcal{A} -comodule algebras. We prove that it is a sheaf of Hopf-Galois extensions; in view of Lemma 4.1 (see the discussion after Definition 4.3) it suffices to show that the global sections are a Hopf-Galois extension.

Proposition 4.8. *The subalgebra of coinvariants $\mathcal{B} := (\mathcal{P}(S^4))^{co(SU(2))}$ with respect to the coaction in (4.10) is*

$$\mathcal{B} = \{(x \otimes 1, y \otimes 1) \in \mathcal{P}(S^4) \mid \rho_{N,SN}(x) = \rho_{S,SN}(y)\} = \mathcal{O}_{S_q^4}(S^4) . \quad (4.11)$$

The extension $\mathcal{B} \subset \mathcal{P}(S^4)$ is Hopf-Galois. Furthermore $\mathcal{P}(S^4)$ is a faithfully flat \mathcal{B} -module.

Proof. The explicit form of \mathcal{B} follows from the definition of the coaction (4.10). To show that $\mathcal{B} \subset \mathcal{P}(S^4)$ is Hopf-Galois we exhibit a strong connection $l : \mathcal{A} \rightarrow \mathcal{P}(S^4) \otimes \mathcal{P}(S^4)$; we set

$$l(h) = (1 \otimes S(h_{(1)}), 0) \otimes (1 \otimes h_{(2)}) + (0, 1 \otimes S(h_{(1)})) \otimes (0, 1 \otimes h_{(2)}). \quad (4.12)$$

We check the three properties l has to satisfy (see (i), (ii) and (iii) before Lemma 4.1). We begin with $(l \otimes \text{id})\Delta = (\text{id} \otimes \delta_{\mathcal{A}})l$. The left hand side reads

$$\begin{aligned} (l \otimes \text{id})\Delta(h) &= l(h_{(1)}) \otimes l(h_{(2)}) \\ &= \left((1 \otimes S(h_{(1)}), 0) \otimes (1 \otimes h_{(2)}) + (0, 1 \otimes S(h_{(1)})) \otimes (0, 1 \otimes h_{(2)}) \right) \otimes h_{(3)} \end{aligned}$$

which agrees with the right hand side

$$(\text{id} \otimes \delta_{\mathcal{A}})l(h) = (\text{id} \otimes \delta_{\mathcal{A}}) \left((1 \otimes S(h_{(1)}), 0) \otimes (1 \otimes h_{(2)}) + (0, 1 \otimes S(h_{(1)})) \otimes (0, 1 \otimes h_{(2)}) \right)$$

once the explicit form of the coaction (4.10) is taken into account. Similarly for $(\text{id} \otimes l)\Delta = (\mathcal{A}\delta \otimes \text{id})l$, where the left hand side is computed as

$$\begin{aligned} (\text{id} \otimes l)\Delta(h) &= h_{(1)} \otimes l(h_{(2)}) \\ &= h_{(1)} \otimes \left((1 \otimes S(h_{(2)}), 0) \otimes (1 \otimes h_{(3)}) + (0, 1 \otimes S(h_{(2)})) \otimes (0, 1 \otimes h_{(3)}) \right) \end{aligned}$$

and the right hand side as

$$\begin{aligned} (\mathcal{H}\delta \otimes \text{id})l(h) &= (\mathcal{A}\delta \otimes \text{id}) \left((1 \otimes S(h_{(1)}), 0) \otimes (1 \otimes h_{(2)}) + (0, 1 \otimes S(h_{(1)})) \otimes (0, 1 \otimes h_{(2)}) \right) \\ &= S^{-1}(S(h_{(1)}))_{(2)} \otimes \left((1 \otimes (S(h_{(1)}))_{(1)}), 0 \right) \otimes (1 \otimes h_{(2)}, 0) + (0, 1 \otimes (S(h_{(1)}))_{(1)}) \otimes (0, 1 \otimes h_{(2)}) \\ &= S^{-1}(S(h_{(1)})) \otimes \left((1 \otimes S(h_{(2)}), 0) \otimes (1 \otimes h_{(3)}, 0) + (0, 1 \otimes S(h_{(2)})) \otimes (0, 1 \otimes h_{(3)}) \right). \end{aligned}$$

Finally, $\tilde{\chi} \circ l = 1 \otimes \text{id}$ is verified as

$$\begin{aligned} \tilde{\chi}(l(h)) &= \tilde{\chi} \left((1 \otimes S(h_{(1)}), 0) \otimes (1 \otimes h_{(2)}) + (0, 1 \otimes S(h_{(1)})) \otimes (0, 1 \otimes h_{(2)}) \right) \\ &= \left((1 \otimes S(h_{(1)}), 0) \cdot (1 \otimes h_{(2)}) + (0, 1 \otimes S(h_{(1)})) \cdot (0, 1 \otimes h_{(2)}) \right) \otimes h_{(3)} \\ &= \left((1 \otimes 1, 0) + (0, 1 \otimes 1) \right) \otimes \varepsilon(h_{(1)}) h_{(2)} = (1 \otimes 1, 1 \otimes 1) \otimes h. \end{aligned}$$

The last assertion then follows from the property of $\mathcal{A}(SU(2))$ to be cosemisimple. \square

4.1. The instanton sheaf. Let us consider the fundamental left corepresentation of $SU(2)$, $\rho : \mathbb{C}^2 \rightarrow \mathcal{A} \otimes \mathbb{C}^2$, $(z_1, z_2) \mapsto A \dot{\otimes} (z_1, z_2)$, where A is the defining matrix introduced in (3.5) and $\dot{\otimes}$ indicates tensor product and matrix multiplication combination. We will use the notation $\rho(z) = z_{(-1)} \otimes z_{(0)}$ for the left coaction ρ on $z = (z_1, z_2) \in \mathbb{C}^2$. For each open set U of S^4 on which the sheaf \mathcal{P} of the quantum Hopf bundle trivializes, the algebra $\mathbb{C}^2 \otimes \mathcal{P}(U)$ can be endowed with a right \mathcal{A} -comodule structure via

$$\Psi : \mathbb{C}^2 \otimes \mathcal{P}(U) \rightarrow \mathbb{C}^2 \otimes \mathcal{P}(U) \otimes \mathcal{A}, \quad z \otimes x \mapsto z_{(0)} \otimes x_{(0)} \otimes S^{-1}(z_{(-1)})x_{(1)}. \quad (4.13)$$

Set $\mathcal{V}(U) := (\mathbb{C}^2 \otimes \mathcal{P}(U))^{\text{co}\mathcal{A}} = \{z \otimes x \mid \Psi(z \otimes x) = z \otimes x \otimes 1\} \subseteq \mathbb{C}^2 \otimes \mathcal{P}(U)$ with algebra structure inherited from $\mathbb{C}^2 \otimes \mathcal{P}(U)$. The assignment $U \mapsto \mathcal{V}(U)$ (and restriction maps given by extending those of \mathcal{P} to the tensor product via the identity on \mathbb{C}^2) defines a sheaf of algebras \mathcal{V} on S^4 . In agreement with [12, §4.2] we refer to \mathcal{V} as the associated quantum vector bundle to \mathcal{P} with typical fiber \mathbb{C}^2 . The following result provides the usual equivalent characterization of associated bundles in terms of base and typical fiber.

Proposition 4.9. *For U as above, there exists an isomorphism of algebras $\mathcal{V}(U) \simeq \mathcal{O}_{S_q^4}(U) \otimes \mathbb{C}^2$.*

Proof. For $U = U_N$ consider the map

$$\begin{aligned} \Gamma_N : \mathcal{O}_{S_q^4}(U_N) \otimes \mathbb{C}^2 &\rightarrow \mathcal{V}(U_N) = (\mathbb{C}^2 \otimes \mathcal{P}(U_N))^{\text{co}\mathcal{A}} \\ a_N \otimes z &\mapsto z_{(0)} \otimes \Omega_N(a_N \otimes z_{(-1)}). \end{aligned} \quad (4.14)$$

Since Ω_N (see (4.8)) is a morphism of right \mathcal{A} -comodules, Γ_N takes indeed values in the \mathcal{A} -coinvariant subalgebra of $\mathbb{C}^2 \otimes \mathcal{P}(U_N)$. The inverse of Γ_N on a generic $(z \otimes (b^N \oplus b^{SN})) \in (\mathbb{C}^2 \otimes \mathcal{P}(U_N))^{\text{co}\mathcal{A}}$ is proven to be $p_1(\Omega_N^{-1}(b^N \oplus b^{SN})) \otimes z$, where p_1 is the projection onto the first factor. For $U = U_S$ the construction is similar. \square

The quantum principal bundle and quantum associated bundle discussed so far reduce to the classical Hopf bundle and associated instanton bundle with topological charge, or instanton number, equal to 1. We recall that classically the instanton number can be characterized as the degree $k \in \pi_3(S^3) = \mathbb{Z}$ of the transition map $\tau_{NS} : SU(2) \simeq S^3 \rightarrow U_N \cap U_S \simeq S^3 \times I \sim S^3$, where \sim (resp. \simeq) stands for homotopic (resp. topological) equivalence. Quantum bundles with higher instanton numbers are obtained via transition functions of higher (topological) degree; for $k \in \mathbb{N}$ we set $\tau_{NN}^k := \tau_{NN}$ and $\tau_{SS}^k := \tau_{SS}$, thus as in (4.3), while

$$\begin{aligned} \tau_{NS}^k : \mathcal{A} &\rightarrow \mathcal{A} \otimes_{\Psi} \mathcal{I}, & \tau_{SN}^k : \mathcal{A} &\rightarrow \mathcal{A} \otimes_{\Psi} \mathcal{I} \\ h &\mapsto h_{(1)} \cdot \dots \cdot h_{(k)} \otimes 1, & h &\mapsto S(h_{(k)}) \cdot \dots \cdot S(h_{(1)}) \otimes 1. \end{aligned}$$

The resulting principal and associated bundles are referred to as the quantum $SU(2)$ -instanton bundles on S_q^4 with charge $k \in \mathbb{N}$. One gets negative charges by exchanging τ_{NS}^k with τ_{SN}^k . For $q = 1$ they reduce to the classical $SU(2)$ -instanton bundles on S^4 of the corresponding charge.

5. THE CONNECTION AND ITS ANTI-SELF DUAL CURVATURE

We define an anti-selfdual connection on the $SU(2)$ -Hopf bundle on S_q^4 . As in many other constructions in this paper, the advantage of a sheaf theoretic approach is that to describe global objects on S_q^4 one can work locally on the two patches U_N and U_S . The local data is then assembled together using restriction maps written in terms of the isomorphism \mathcal{Q} , as done for example in (3.24). We present explicit formulas for U_N only, the case of U_S being completely similar.

Let us consider the following two one-forms in ${}_N\Omega_q^\bullet$:

$$\eta_1 := \beta_{23}^* d\beta_{23} + q^2 \beta_{24} d\beta_{24}^* - d\beta_{23}^* \beta_{23} - q^2 d\beta_{24} \beta_{24}^* \quad (5.1)$$

$$\eta_2 := 2(\beta_{23}^* d\beta_{24} - q^2 \beta_{24} d\beta_{23}^*). \quad (5.2)$$

We derive some identities to be used shortly after.

Lemma 5.1. $\eta_1 \wedge_q \eta_1 = 0$.

Proof. To perform the computation we chose the following (arbitrary) order among zero and one forms:

$$\beta_{23}^* < \beta_{23} < \beta_{24} < \beta_{24}^* < d\beta_{23}^* < d\beta_{23} < d\beta_{24} < d\beta_{24}^*$$

then, by using the commutation relations in ${}_N\Omega_q^\bullet$ and omitting the \wedge_q mark we have

$$\begin{aligned}
\eta_1 \wedge_q \eta_1 &= q^2 \beta_{23}^* d\beta_{23} \beta_{24} d\beta_{24}^* - \beta_{23}^* d\beta_{23} d\beta_{23}^* \beta_{23} - q^2 \beta_{23}^* d\beta_{23} d\beta_{24} \beta_{24}^* + q^2 \beta_{24} d\beta_{24}^* \beta_{23}^* d\beta_{23} + \\
&\quad - q^2 \beta_{24} d\beta_{24}^* d\beta_{23}^* \beta_{23} - q^4 \beta_{24} d\beta_{24}^* d\beta_{24} \beta_{24}^* - d\beta_{23}^* \beta_{23} \beta_{23}^* d\beta_{23} - q^2 d\beta_{23}^* \beta_{23} \beta_{24} d\beta_{24}^* + \\
&\quad q^2 d\beta_{23}^* \beta_{23} d\beta_{24} \beta_{24}^* - q^2 d\beta_{24} \beta_{24}^* \beta_{23}^* d\beta_{23} - q^4 d\beta_{24} \beta_{24}^* \beta_{24} d\beta_{24}^* + q^2 d\beta_{24} \beta_{24}^* d\beta_{23}^* \beta_{23} \\
&= q^2 \beta_{23}^* \beta_{24} d\beta_{23} d\beta_{24}^* + \beta_{23}^* \beta_{23} d\beta_{23}^* d\beta_{23} - q^{-2} \beta_{23}^* \beta_{24}^* d\beta_{23} d\beta_{24} - q^2 \beta_{23}^* \beta_{24} d\beta_{23} d\beta_{24}^* + \\
&\quad q^6 \beta_{23} \beta_{24} d\beta_{23}^* d\beta_{24}^* + q^4 \beta_{24} \beta_{24}^* d\beta_{24} d\beta_{24}^* - \beta_{23}^* \beta_{23} d\beta_{23}^* d\beta_{23} - q^6 \beta_{23} \beta_{24} d\beta_{23}^* d\beta_{24}^* + \\
&\quad q^2 \beta_{23} \beta_{24}^* d\beta_{23}^* d\beta_{24} + q^{-2} \beta_{23}^* \beta_{24}^* d\beta_{23} d\beta_{24} - q^4 \beta_{24} \beta_{24}^* d\beta_{24} d\beta_{24}^* - q^2 \beta_{23} \beta_{24}^* d\beta_{23}^* d\beta_{24} \\
&= 0
\end{aligned}$$

□

Lemma 5.2. $\eta_1 \wedge_q \eta_2 = -\eta_2 \wedge_q \eta_1$.

Proof. On the one hand

$$\begin{aligned}
\frac{1}{2} \eta_1 \wedge_q \eta_2 &= \beta_{23}^* d\beta_{23} \beta_{23}^* d\beta_{24} - q^2 \beta_{23}^* d\beta_{23} \beta_{24} d\beta_{24}^* + q^2 \beta_{24} d\beta_{24}^* \beta_{23}^* d\beta_{24} - q^4 \beta_{24} d\beta_{24}^* \beta_{24} d\beta_{23}^* \\
&\quad - q^2 \beta_{23} d\beta_{23}^* \beta_{23}^* d\beta_{24} + q^2 \beta_{24}^* d\beta_{24} \beta_{24} d\beta_{23}^* \\
&= q^{-2} \beta_{23}^* \beta_{23}^* d\beta_{23} d\beta_{24} + \beta_{23}^* \beta_{24} d\beta_{23}^* d\beta_{23} - q^2 \beta_{23}^* \beta_{24} d\beta_{24} d\beta_{24}^* + q^6 \beta_{24} \beta_{24} d\beta_{23}^* d\beta_{24}^* \\
&\quad - \beta_{23}^* \beta_{23} d\beta_{23}^* d\beta_{24} - q^2 \beta_{24} \beta_{24}^* d\beta_{23}^* d\beta_{24}
\end{aligned}$$

On the other hand

$$\begin{aligned}
\frac{1}{2} \eta_2 \wedge_q \eta_1 &= \beta_{23}^* d\beta_{24} \beta_{23}^* d\beta_{23} + q^2 \beta_{23}^* d\beta_{24} \beta_{24} d\beta_{24}^* - q^2 \beta_{23}^* d\beta_{24} \beta_{23} d\beta_{23}^* - q^2 \beta_{24} d\beta_{23}^* \beta_{23}^* d\beta_{23} \\
&\quad - q^4 \beta_{24} d\beta_{23}^* \beta_{24} d\beta_{24}^* + q^2 \beta_{24} d\beta_{23}^* \beta_{24}^* d\beta_{24} \\
&= -q^{-2} \beta_{23}^* \beta_{23}^* d\beta_{23} d\beta_{24} + q^2 \beta_{23}^* \beta_{24} d\beta_{24} d\beta_{24}^* + \beta_{23}^* \beta_{23} d\beta_{23}^* d\beta_{24} - \beta_{23}^* \beta_{24} d\beta_{23}^* d\beta_{23} \\
&\quad - q^6 \beta_{24} \beta_{24} d\beta_{23}^* d\beta_{24}^* + q^2 \beta_{24} \beta_{24}^* d\beta_{23}^* d\beta_{24}
\end{aligned}$$

so that $\eta_1 \wedge_q \eta_2 = -\eta_2 \wedge_q \eta_1$.

□

Lemma 5.3. $\eta_2 \wedge_q \eta_2^* = -\eta_2^* \wedge_q \eta_2$.

Proof. Similarly to above, with some algebra we compute

$$\begin{aligned}
\frac{1}{4} \eta_2 \wedge_q \eta_2^* &= q^2 \beta_{23}^* \beta_{23} d\beta_{24} d\beta_{24}^* + q^{-2} \beta_{23}^* \beta_{24}^* d\beta_{23} d\beta_{24} - q^6 \beta_{23} \beta_{24} d\beta_{23}^* d\beta_{24}^* + q^2 \beta_{24} \beta_{24}^* d\beta_{23}^* d\beta_{23} \\
&= -\frac{1}{4} \eta_2^* \wedge_q \eta_2
\end{aligned}$$

□

Observe that in ${}_N\Omega_q^\bullet$ we have

$$\eta_1 r^2 = r^2 \eta_1, \quad \eta_2 r^2 = r^2 \eta_2$$

for $r^2 = \beta_{24}^* \beta_{24} + \beta_{23}^* \beta_{23}$ (cf. (3.3)). Let us extend the algebra ${}_N\Omega_q^\bullet$ by a generator t and its differential $d(t) = dt$ and quotient by the relation

$$t(1 + r^2) = 1 = (1 + r^2)t.$$

By imposing the Leibniz rule, from the previous equation we get $dt = -t^2 d(r^2)$. Furthermore, since $r^2 \eta_i = \eta_i r^2$ it follows that t (and dt) has to commute with the η_i , $i = 1, 2$.

Consider now the matrix

$$A := \frac{1}{2}t \begin{pmatrix} \eta_1 & \eta_2 \\ -\eta_2^* & \eta_1^* \end{pmatrix} \quad (5.3)$$

We observe that $\eta_1^* = -\eta_1$ and $A \in su(2) \otimes {}_N\Omega_q^1$, so that A is the local restriction to U_N of a connection one-form on the quantum Hopf bundle.

By using the Lemmas above, the curvature $F_A = dA + A \wedge_q A$ of the $SU(2)$ potential \mathcal{A} reduces to

$$F_A = \frac{1}{2} \begin{pmatrix} -t^2 dr^2 \wedge_q \eta_1 + t d\eta_1 - \frac{1}{2}t^2 \eta_2 \wedge_q \eta_2^* & -t^2 dr^2 \wedge_q \eta_2 + t d\eta_2 + t^2 \eta_1 \wedge_q \eta_2 \\ t^2 dr^2 \wedge_q \eta_2^* - t d\eta_2^* - t^2 \eta_2^* \wedge_q \eta_1 & t^2 dr^2 \wedge_q \eta_1 - t d\eta_1 + \frac{1}{2}t^2 \eta_2 \wedge_q \eta_2^* \end{pmatrix} \quad (5.4)$$

and F_A is an $su(2)$ -valued two-form on ${}_N\mathbb{R}_q^4$ as expected.

Theorem 5.4. *The curvature F_A has the expression*

$$F_A = t^2 \begin{pmatrix} d\beta_{23}^* d\beta_{23} + q^2 d\beta_{24} d\beta_{24}^* & 2d\beta_{23}^* d\beta_{24} \\ -2d\beta_{24}^* d\beta_{23} & -d\beta_{23}^* d\beta_{23} - q^2 d\beta_{24} d\beta_{24}^* \end{pmatrix} \quad (5.5)$$

and it is anti-selfdual, $\star_q F_A = -F_A$.

Proof. We start by computing the single summand of the entrance $(F_A)_{11}$:

$$\begin{aligned} -\frac{1}{2}t^2 dr^2 \wedge_q \eta_1 &= -\frac{1}{2}t^2 \left(\beta_{23}^* \beta_{23} d\beta_{23}^* d\beta_{23} + q^6 \beta_{23} \beta_{24} d\beta_{23}^* d\beta_{24}^* - q^2 \beta_{23} \beta_{24}^* d\beta_{23}^* d\beta_{24} + q^2 \beta_{23}^* \beta_{24} d\beta_{23} d\beta_{24}^* \right. \\ &\quad \left. + \beta_{23}^* \beta_{23} d\beta_{23}^* d\beta_{23} - q^{-2} \beta_{23}^* \beta_{24}^* d\beta_{23} d\beta_{24} - q^2 \beta_{23}^* \beta_{24} d\beta_{23} d\beta_{24}^* + q^6 \beta_{23} \beta_{24} d\beta_{23}^* d\beta_{24}^* \right. \\ &\quad \left. + q^4 \beta_{24} \beta_{24}^* d\beta_{24} d\beta_{24}^* - q^{-2} \beta_{23}^* \beta_{24}^* d\beta_{23} d\beta_{24} + q^4 \beta_{24} \beta_{24}^* d\beta_{24} d\beta_{24}^* + q^2 \beta_{23} \beta_{24}^* d\beta_{23}^* d\beta_{24} \right) \\ &= -t^2 \left(\beta_{23}^* \beta_{23} d\beta_{23}^* d\beta_{23} + q^6 \beta_{23} \beta_{24} d\beta_{23}^* d\beta_{24}^* - q^{-2} \beta_{23}^* \beta_{24}^* d\beta_{23} d\beta_{24} + q^4 \beta_{24} \beta_{24}^* d\beta_{24} d\beta_{24}^* \right) \end{aligned}$$

The term involving the differential of η_1 is $\frac{1}{2}t d\eta_1 = t(d\beta_{23}^* d\beta_{23} + q^2 d\beta_{24} d\beta_{24}^*)$ while $\eta_2 \wedge_q \eta_2^*$ was already computed in Lemma 5.3 and gives

$$-\frac{1}{4}t^2 \eta_2 \wedge_q \eta_2^* = -t^2 \left(q^2 \beta_{23}^* \beta_{23} d\beta_{24} d\beta_{24}^* + q^{-2} \beta_{23}^* \beta_{24}^* d\beta_{23} d\beta_{24} - q^6 \beta_{23} \beta_{24} d\beta_{23}^* d\beta_{24}^* + q^2 \beta_{24} \beta_{24}^* d\beta_{23}^* d\beta_{23} \right).$$

Summing them we obtain

$$\begin{aligned} (F_A)_{11} &= \left(-t^2 \beta_{23}^* \beta_{23} + t - q^2 t^2 \beta_{24} \beta_{24}^* \right) (d\beta_{23}^* d\beta_{23} + q^2 d\beta_{24} d\beta_{24}^*) \\ &= \left(-t^2 r^2 + t \right) (d\beta_{23}^* d\beta_{23} + q^2 d\beta_{24} d\beta_{24}^*) \\ &= t^2 (d\beta_{23}^* d\beta_{23} + q^2 d\beta_{24} d\beta_{24}^*) \end{aligned}$$

Similarly, we now compute $(F_A)_{12}$:

$$\begin{aligned} -\frac{1}{2}t^2 dr^2 \wedge_q \eta_2 &= -t^2 \left(\beta_{23}^* \beta_{23} d\beta_{23}^* d\beta_{24} + q^{-2} \beta_{23}^* \beta_{23}^* d\beta_{23} d\beta_{24} + \beta_{23}^* \beta_{24} d\beta_{23}^* d\beta_{23} - q^2 \beta_{23}^* \beta_{24} d\beta_{24} d\beta_{24}^* + \right. \\ &\quad \left. + q^6 \beta_{24} \beta_{24} d\beta_{23}^* d\beta_{24}^* + q^2 \beta_{24} \beta_{24}^* d\beta_{23}^* d\beta_{24} \right) \\ \frac{1}{2}td\eta_2 &= t(d\beta_{23}^* d\beta_{24} - q^2 d\beta_{24} d\beta_{23}^*) = 2td\beta_{23}^* d\beta_{24}. \end{aligned}$$

We sum them to $\frac{1}{2}t^2\eta_1 \wedge_q \eta_2$, whose expression was computed in 5.2, and the only terms which do not cancel give

$$(F_A)_{12} = \left(-2t^2 \beta_{23}^* \beta_{23} + 2t - 2q^2 t^2 \beta_{24} \beta_{24}^* \right) d\beta_{23}^* d\beta_{24} = \left(-2t^2 r^2 + 2t \right) d\beta_{23}^* d\beta_{24} = 2t^2 d\beta_{23}^* d\beta_{24}.$$

Recalling the explicit expression of anti-selfdual forms in ${}_N\Omega_q^\bullet$, see (3.32), we conclude that F_A is anti-selfdual. \square

The above Theorem shows that the connection A in (5.3)(together with its analogue in ${}_S\Omega_q^\bullet$) describes an anti-instanton on $\mathcal{A}(S_q^4)$. For $q = 1$ the potential A reduces to 't Hooft basic instanton ([1, Ch.II]) and its curvature F_A agrees with $F_A = (1 + |\mathbf{x}|^2)^{-2} d\bar{x}dx$ in quaternionic notation $\mathbf{x} = x_0 + i x_1 + j x_2 + k x_3$ (for the relation among generators β and x_i see (3.31)).

APPENDIX A. TWISTED TENSOR PRODUCTS

Consider two associative and unital algebras (A, \cdot_A) and (B, \cdot_B) . Let $a, a' \in A, b, b' \in B$. A twist map is a linear map $\Psi : B \otimes A \rightarrow A \otimes B$, $\Psi(b \otimes a) = a^{[\Psi]} \otimes b^{[\Psi]}$. To each twist map Ψ one can associate a multiplication

$$\cdot_\Psi = (\cdot_A \otimes \cdot_B)(\text{id}_A \otimes \Psi \otimes \text{id}_B)$$

in the tensor product $A \otimes B$. Algebras of the form $(A \otimes B, \cdot_\Psi)$ are referred to as twisted tensor product algebras $A \otimes_\Psi B$. The ordinary tensor product algebra $(A \otimes B, \cdot_\otimes)$ corresponds to $\Psi = \tau$, the flip map. A twist map is said to be normal if $\Psi(b \otimes 1_A) = 1_A \otimes b$ and $\Psi(1_B \otimes a) = a \otimes 1_B$ for every a, b . There are necessary and/or sufficient conditions on Ψ such that \cdot_Ψ is unital and/or associative, see e.g. [2] for normal twists and [5] for non-normal twists.

We are interested in the compatibility of twisted tensor products with $*$ -structures. Let A, B unital $*$ -algebras with $*$ -structures $*_A, *_B$ respectively. Let $*_\otimes := (*_A \otimes *_B)$.

Definition A.1. A map $\Psi' : A \otimes B \rightarrow A \otimes B$ is said to be $*$ -compatible provided

$$*_\otimes = \Psi' \circ *_\otimes \circ \Psi' \quad (\text{A.1})$$

Proposition A.2. Let $\Psi : B \otimes A \rightarrow A \otimes B$ be a normal twist. If $\Psi' := \Psi \circ \tau$ is $*$ -compatible, then

$$*_\Psi := (\Psi \circ \tau) \circ *_\otimes : a \otimes b \mapsto \Psi(b^* \otimes a^*) \quad (\text{A.2})$$

defines a $*$ -structure in the twisted tensor algebra $A \otimes_\Psi B$.

Proof. It follows immediately from (A.1) that $*_\Psi \circ *_\Psi = \text{id}$. To prove that $*_\Psi$ is an algebra involution we proceed in different steps. We denote by \cdot_Ψ the multiplication in $A \otimes_\Psi B$. First, by using (A.1) and the fact that Ψ is normal, we have

$$((1 \otimes b) \cdot_\Psi (a \otimes 1))^* = \Psi' \circ *_\otimes (\Psi(b \otimes a)) = *_\otimes (a \otimes b) = (a \otimes 1)^* \cdot_\Psi (1 \otimes b)^*.$$

Similarly, from the definition of $*_{\Psi}$, we can prove the results on $(a \otimes 1) \cdot_{\Psi} (1 \otimes b)$. Finally, before to prove the result in full generality, we show it holds on $(a \otimes 1) \cdot_{\Psi} (a' \otimes b)$ and $(a \otimes b) \cdot_{\Psi} (1 \otimes b')$. For this we need to use the eqs. (3.8) and (3.9). For instance

$$\begin{aligned} ((a \otimes 1) \cdot_{\Psi} (a' \otimes b))^{*\Psi} &= \Psi(b^* \otimes a'^* a^*) = (a'^*)^{[\Psi]} (a^*)^{[[\Psi]]} \otimes (b^*)^{[\Psi]} [[\Psi]] \\ &= \left((a'^*)^{[\Psi]} \otimes (b^*)^{[\Psi]} \right) \cdot_{\Psi} (a^* \otimes 1) = (a' \otimes b)^{*\Psi} \cdot_{\Psi} (a \otimes 1)^{*\Psi} \end{aligned}$$

where in the second equality we have used (3.9). Then, we can conclude that $*_{\Psi}$ is an involution on the generic element $(a \otimes b) \cdot_{\Psi} (a' \otimes b')$ by using the hypothesis of normality of Ψ to split the product as $(a \otimes b) \cdot_{\Psi} (a' \otimes b') = (a \otimes 1) \cdot_{\Psi} (1 \otimes b) \cdot_{\Psi} (a' \otimes 1) \cdot_{\Psi} (1 \otimes b')$ and thus by applying the above intermediate results. \square

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